

Danzer's configuration revisited

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Dedicated to the memory of Ludwig Danzer

Abstract

We revisit the configuration of Danzer $DCD(4)$, a great inspiration for our work. This configuration of type (35_4) falls into an infinite series of geometric point-line configurations $DCD(n)$. Each $DCD(n)$ is characterized combinatorially by having the Kronecker cover over the Odd graph O_n as its Levi graph. Danzer's configuration is deeply rooted in Pascal's Hexagrammum Mysticum. Although the combinatorial configuration is highly symmetric, we conjecture that there are no geometric point-line realizations with 7- or 5-fold rotational symmetry; on the other hand, we found a point-circle realization having the symmetry group D_7 , the dihedral group of order 14.

Keywords: Danzer configuration, Danzer graph, Odd graph, Kronecker cover, V-construction, Hexagrammum Mysticum, point-circle configuration, Cayley-Salmon configuration, Steiner-Plücker configuration, Coxeter (28_3) -configuration.

Math. Subj. Class.: 51A20, 52C30, 52C35, 05B30.

1 Introduction

There is a remarkable point-line configuration of type (35_4) , due to Ludwig Danzer, the first known geometric (n_4) configuration with n not divisible by 3, which he never published [21, 23]. It has many interesting combinatorial, geometric and graph-theoretic relationships, and our goal in this paper is to present and discuss some of them. To put this object into a proper historical context, we should go back to Arthur Cayley, 1846. Cayley has given the following construction for Desargues' (10_3) configuration [10]. Let P_1, \dots, P_5 be five points in projective 3-space in general position, i.e. such that no four of them lie in a common plane. Then the 10 lines $P_i P_j$ and the 10 planes $P_i P_j P_k$ form a line-plane configuration such that its intersection with a plane containing none of the points P_i yields the point-line (10_3) configuration of Desargues. (We note that the same construction is also used, without explicit mention of Cayley, by Levi in his book [27].) Take now the dual construction. Instead of points, in this case we start from five planes in general position (i.e. no more than two planes meet in a line). Then every three of them meet in a point, and every two of them meet in a line, forming directly the (10_3) point-line Desargues configuration.

Danzer's construction is completely analogous. It uses seven 3-spaces in general position in the 4-dimensional projective space. They meet by fours and by threes in 35 points and 35 lines, respectively; hence we obtain a (35_4) configuration. (By suitable projection it can be carried over to a corresponding planar configuration).

The possibility of a common generalization of Cayley's and Danzer's construction is now clear. Consider an n -dimensional projective space ($n \geq 2$). In this space, take $2n - 1$ hyperplanes in general position; this means that no more than n of them meet in a point. We shall call such an arrangement of hyperplanes in a projective n -space a *Cayley-Danzer arrangement*. Now n hyperplanes of this arrangement meet in a point, and $n - 1$ hyperplanes meet in a line. Thus we have altogether $\binom{2n-1}{n}$ points and $\binom{2n-1}{n-1}$ lines (hence their number is the same). These points and lines can be considered as labelled with the n - and $(n - 1)$ -element subsets, respectively, of a $(2n - 1)$ -set. Moreover, the incidence between the points and lines is determined by containment between the corresponding subsets. Hence it is clear that each point is incident with n lines, and vice versa. Thus, after projection to a suitable plane, we have a planar point-line configuration of type

$$\left(\binom{2n-1}{n-1}_n \right). \quad (1)$$

Thus we see that both Desargues' (10_3) and Danzer's (35_4) configuration are

particular cases of (1); therefore, the class of configurations described above may be termed a *Desargues-Cayley-Danzer configuration*. We shall denote it by $DCD(n)$.

In this paper we first discuss further interesting historical connections of Danzer's configuration (Section 2). Namely, we show that in fact it was already known in the nineteenth century, as a member of the infinite family of configurations associated with Pascal's *Hexagrammum Mysticum*. Then, in Section 3, we explore some graph-theoretic connections of the configurations $DCD(n)$. In particular, we show that each $DCD(n)$ is characterized combinatorially by having the Kronecker cover over the Odd graph O_n as its Levi graph. In Section 4 we present a decomposition of $DCD(n)$. In addition to the subconfigurations found in this way, in Section 5 we establish the existence of another subconfiguration which is closely related to the well-known Coxeter graph. All these subconfigurations can be used to find explicit constructions for a graphical representation of Danzer's configuration. Section 6 is devoted to this task. Finally, in Section 7 we show that all the configurations studied in the former sections have not only point-line, but also point-circle representations. In particular, we show that although $DCD(4)$ seems to admit no point-line representation with 7-fold symmetry, there are point-circle examples realizing D_7 symmetry.

Here we recall that a *configuration*, in general, consists of p points and n lines, such that k of the points lie on each line and q of the lines pass through each point. (Instead of lines, one can also take other geometric figures, such as e.g. circles. Moreover, a configuration can also be defined as a combinatorial incidence structure consisting of p "points" and n "blocks", such that these terms have no specific meaning; this structure is termed an *combinatorial configuration*.) Its usual notation is (p_q, n_k) . If, in particular, $p = n$, then $q = k$; in this case the notation (n_k) is used, and we speak of a *balanced configuration* [22].

2 The (35_4) configuration associated with the complete Pascal hexagon

Pascal's famous theorem states that if a hexagon is inscribed in a conic, then the three pairs of opposite sides meet in three points on a straight line. This line is called the *Pascal line*. Taking the permutations of six points on a conic, one obtains 60 different hexagons. Thus, the so-called *complete Pascal hexagon* determines altogether 60 Pascal lines. Jacob Steiner started to investigate the 60 Pascal lines, and he found that these lines meet in triples in 20 points (1828 [26, 33, 9]). These latter, which are now called *Steiner*

points, lie, in turn, in quadruples on 15 *Plücker lines* (Plücker, 1929, see [9], and 1830, see [26]). Thus we get a $(20_3, 15_4)$ configuration, which we call the *Steiner-Plücker configuration*. Here we have the following incidence theorem (see Exercise II.16.6 in [36] and Exercise 2.3.2 in [12]).

Theorem 2.1. *If three triangles are perspective from the same point, the three axes of perspectivity of the three pairs of triangles are concurrent.*

By Hesse, the $(20_3, 15_4)$ Steiner-Plücker configuration coincides with the configuration formed by the points and lines in this theorem (1850, see [26]). (For an illustration, see Figure 4 in Section 4.)

On the other hand, Kirkman discovered that the 60 Pascal lines intersect in threes in 60 additional points; in fact, the Pascal lines together with these *Kirkman points* form a (60_3) configuration (1849 [26, 27, 9]). Cayley showed that the Kirkman points lie in triples on 20 new lines (1849 [26, 9]). These 20 *Cayley lines* were found to be concurrent in triples in 15 *Salmon points* (Salmon, 1849, see [26, 9]). The 20 Cayley lines and the 15 Salmon points form a $(15_4, 20_3)$ configuration. Here we have again an incidence theorem, which is dual to the former ([36], Exercise II.16.6)

Theorem 2.2. *If three triangles are perspective from the same line, the three centres of perspectivity of the three pairs of triangles are collinear.*

The *Cayley-Salmon configuration* formed by the 20 Cayley lines and the 15 Salmon points is the same as that obtained from the points and lines of this theorem. Thus the Steiner-Plücker configuration and this latter configuration are dual to each other. It is to be emphasized, however, that in spite of this dual correspondence, there is no polarity (in contrast to Hesse, 1868, see [26]), and not even a general projective correlation (in contrast to Schröter, 1876, see [26]), which would carry the Pascal lines, Steiner points and Plücker lines into the Kirkman points, Cayley lines and Salmon points, respectively (cf. Remark 4.2 later). This was already known to Leopold Klug, who published a comprehensive work in 1898 on the configurations associated with the complete Pascal hexagon [26].

Now, on page 34 of his book, Klug makes the following observation.

Theorem 2.3. *The $(20_3, 15_4)$ Steiner-Plücker configuration and the $(15_4, 20_3)$ Cayley-Salmon configuration together form a (35_4) configuration. The 35 points and the 35 lines of this configuration are the 20 Steiner points and 15 Salmon points, and the 15 Plücker lines and 20 Cayley lines, respectively. On each Plücker line there are four Steiner points, and on each Cayley line there are three Salmon points and one Steiner point; moreover, three Plücker lines and one Cayley line pass through each Steiner point, and four Cayley lines pass through each Salmon point.*

He also remarks that this configuration is nothing else than the planar projection of a figure formed

- by the vertices and edges of three tetrahedra, which are perspective from the same point;
- by the points and lines of intersection of (the projective hulls of) the corresponding edges and faces, respectively;
- and finally, by the projecting lines and by the lines of intersections of planes of perspectivity.

In this latter observation, the *plane of perspectivity* is an analogue of one dimension higher of the axis of perspectivity, as it occurs in the following theorem on perspective tetrahedra ([36], §17, Theorem 2; this theorem is already mentioned in one of Felix Klein's works, 1870, see Carver [7]).

Theorem 2.4. *If two tetrahedra are perspective from a point, the six pairs of lines of the corresponding edges intersect in coplanar points, and the planes of the four pairs of faces intersect in coplanar lines; i.e. the tetrahedra are perspective from a plane.*

We remark that the configurations associated to the complete Pascal hexagon is a vast topic. In fact, there are infinitely many such configurations [26]; we do not pursue this topic further, since here we are only interested in the historical background of the particular case of the (35_4) configuration. For further details, the reader is referred to [2, 26, 27, 28, 33]; in a quite recent contribution, Conway and Ryba [9] throw new light upon the *Hexagrammum Mysticum*, i.e. the system of 95 points and 95 lines consisting of the (60_3) configuration of the Pascal lines and Kirkman points, and our (35_4) configuration.

3 Relationship between $DCD(n)$ and the Odd graph O_n

Recall that the *Kneser graph* $K(n, k)$ has as vertices the k -subsets of an n -element set, where two vertices are adjacent if the k -subsets are disjoint. The Kneser graph $K(2n - 1, n - 1)$ is called an *Odd graph* and is denoted by O_n [6, 18, 19]. By a simple comparison of this definition with that given in Section 1, it is immediately clear that the configuration $DCD(n)$ and the Odd graph O_n are closely related to each other.

This relationship can be understood via the abstract V -construction, introduced and discussed in [17]. Here we recall some definitions and results from that paper.

Let $k \geq 2$, $n \geq 3$ be integers and let G be a regular graph of valency k on n vertices. For a vertex v of G , denote by $N(v)$ the set of vertices adjacent to v . Then take the family $S(G)$ of these vertex-neighbourhoods:

$$S(G) = \{N(v) \mid v \in V(G)\},$$

where $V(G)$ denotes the set of vertices of G . The triple $(V(G), S(G), \in)$ defines a combinatorial incidence structure, which we denote by $N(G)$. We call the graph G *admissible* if no two of its vertices have a common neighbourhood (in [32] a graph with this property is called *worthy*). For an admissible graph G , the incidence structure $N(G)$ is a combinatorial (n_k) configuration. In this case we say that (n_k) is obtained from G by *V-construction*.

Recall that the *Levi graph* $L(C)$ of a configuration C is a bipartite graph whose bipartition classes consist of the points and blocks of C , respectively, and two points in $L(C)$ are adjacent if and only if the corresponding point and block in C are incident. Many properties of configurations can be described by Levi graphs. For instance, a configuration is flag-transitive if and only if its Levi graph is edge-transitive. We have the following classical result [10, 22, 28].

Lemma 3.1. *A configuration C is uniquely determined by its Levi graph $L(C)$ with a given vertex coloring.*

If, in particular, C is obtained by V -construction, the following property is a useful tool in the study of such configurations [17].

Theorem 3.2. *Let G be an admissible graph, and let L be the Levi graph of the configuration C obtained by V -construction from G . Then L is the Kronecker cover of G .*

Here we recall that a graph \tilde{G} is said to be the *Kronecker cover* (or *canonical double cover*) of the graph G if there exists a $2 : 1$ surjective homomorphism $f : \tilde{G} \rightarrow G$ such that for every vertex v of \tilde{G} the set of edges incident with v is mapped bijectively onto the set of edges incident with $f(v)$ [19, 25].

We say that a configuration is *combinatorially self-polar* if there exists an automorphism of order two of its Levi graph interchanging the two parts of bipartition (see e.g. [28]). The following result is also taken from [17].

Theorem 3.3. *A configuration obtained by V -construction from an admissible graph G is combinatorially self-polar.*

Take now the Odd graph O_n , $n \geq 2$, and consider a vertex v of O_n . Since v is an $(n-1)$ -subset of the $(2n-1)$ -element set, its neighbourhood $N(v)$ consists of $(n-1)$ -subsets disjoint to v . Take the union of these latter sets, and denote it by $\bar{N}(v)$. Clearly, $\bar{N}(v)$ is complementary to v in the $(2n-1)$ -element set. It immediately follows that O_n is admissible, hence the V -construction can be applied. It yields a configuration in which the block $\bar{N}(v)$ is incident with a point v' , such that v' is adjacent to v in the graph O_n . Hence the Levi graph of this configuration is a bipartite graph whose “black” and “white” vertices are the n -subsets and $(n-1)$ -subsets of the $(2n-1)$ -element set, respectively, where adjacency is given by containment. This graph is called the *revolving door graph*, or *middle-levels graph* (a subclass of the *bipartite Kneser graphs*; see e.g. [17] and the references therein). On the other hand, it is directly seen that the Levi graph of the configuration $DCD(n)$ is the same graph (consider the Cayley-Danzer arrangement from which it is derived). Hence, by Lemma 3.1, we obtain the following main result of this section.

Theorem 3.4. *For $n \geq 2$, the configuration $DCD(n)$ is isomorphic to the combinatorial configuration obtained by V -construction from the Odd graph O_n .*

A consequence is that this combinatorial configuration has the same type $\left(\binom{2n-1}{n-1}_n\right)$ as given in (1) in Section 1. The configuration $DCD(2)$ corresponds to the trilateral (3_2) , the case $n = 3$ gives rise to the Desargues configuration $DCD(3)$ of type (10_3) , while the particular case $n = 4$ corresponds to Danzer’s configuration (35_4) .

Corollary 3.5. *Danzer’s configuration (35_4) is isomorphic to the combinatorial configuration obtained by V -construction from the Odd graph O_4 .*

A representation of the graph O_4 is depicted in Figure 1.

4 Decomposition of $DCD(n)$

The following definition is taken from [16].

Definition 1. By the *incidence sum* of configurations \mathcal{C}_1 and \mathcal{C}_2 we mean the configuration \mathcal{C} , which is the disjoint union of \mathcal{C}_1 and \mathcal{C}_2 , together with a specified set $I \subseteq \mathcal{P}_1 \times \mathcal{L}_2 \cup \mathcal{P}_2 \times \mathcal{L}_1$ of incident point-line pairs, where \mathcal{P}_i denotes the point set and \mathcal{L}_i denotes the line set of \mathcal{C}_i , for $i = 1, 2$. We denote it by $\mathcal{C}_1 \oplus_I \mathcal{C}_2$.

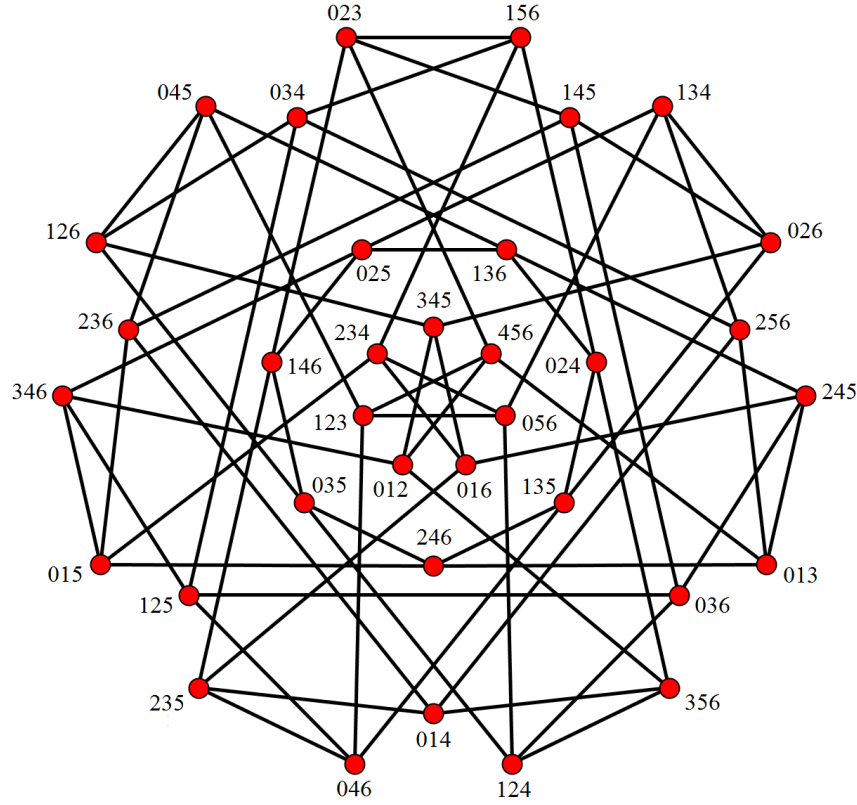


Figure 1: The Odd graph O_4 , realized with a 7-fold dihedral symmetry.

If the set I is clear from the context, it can be omitted from the operation symbol. Figure 2 shows the example that the Desargues configuration can be considered as the incidence sum of a complete quadrangle and a complete quadrilateral (or, in other words, the incidence sum of the *Pasch configuration* and its dual). The set of new incidences consists of 6 point-line pairs formed by the lines of the complete quadrangle and by the points of the complete quadrilateral.

The decomposition can also be observed in the Levi graph of the configuration. In Figure 3 the Desargues graph is presented in such a way that the removal of six independent edges decompose it into two copies of the subdivided complete graph K_4 with opposite coloring. One of them is the Levi graph of the complete quadrangle, while the other one is the Levi graph of the complete quadrilateral.

We show that this is a particular (actually, the second smallest) case of the following general relationship.

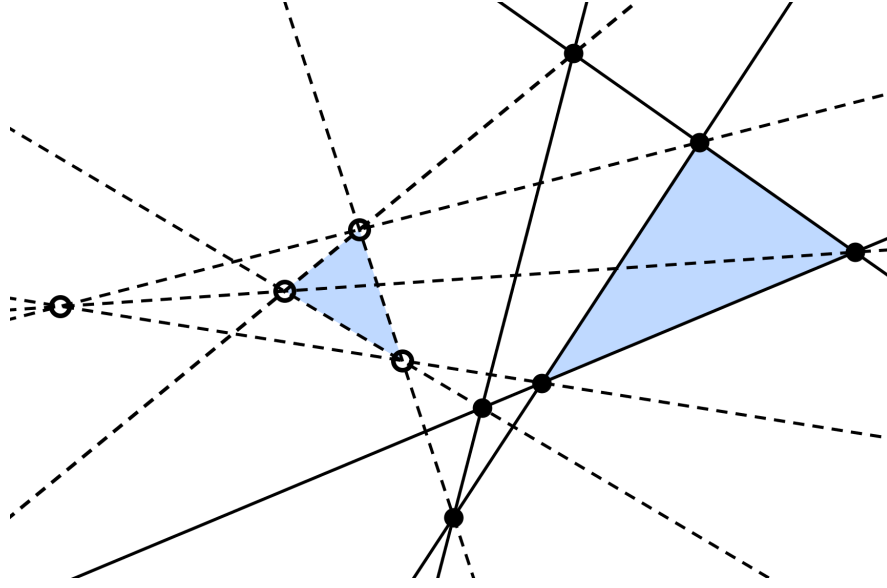


Figure 2: The Desargues configuration (10_3) is the incidence sum of a complete quadrangle $(4_3, 6_2)$ and a complete quadrilateral $(6_2, 4_3)$. (The former denoted by white points and dashed lines, while the latter by black points and solid lines.)

Theorem 4.1. *For all $n \geq 3$, the configuration $DCD(n)$ is the incidence sum of the form $\mathcal{C}_1 \oplus_I \mathcal{C}_2$ such that*

(1) \mathcal{C}_1 is a configuration of type

$$\left(\binom{2n-2}{n-2}_n, \binom{2n-2}{n-1}_{n-1} \right); \quad (2)$$

(2) \mathcal{C}_2 is a configuration of type

$$\left(\binom{2n-2}{n-1}_{n-1}, \binom{2n-2}{n-2}_n \right); \quad (3)$$

(3) the set I of new incidences consists of $\binom{2n-2}{n-1}$ point-line pairs whose points belong to \mathcal{C}_2 and whose lines belong to \mathcal{C}_1 ;

(4) \mathcal{C}_1 and \mathcal{C}_2 are dual to each other;

(5) \mathcal{C}_1 and \mathcal{C}_2 are flag-transitive configurations.

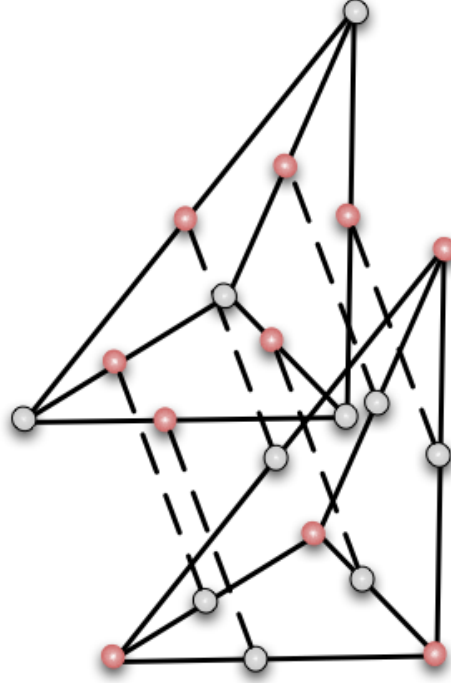


Figure 3: Decomposition of the Desargues graph into two disjoint copies of the subdivided complete graph on 4 vertices.

Proof. Consider first the $\binom{2n-1}{n-1}$ points which originate from the $\binom{2n-1}{n-1} = \binom{2n-1}{n}$ n -tuples of hyperplanes determining $DCD(n)$, as described in Section 1. Using Pascal's rule:

$$\binom{2n-1}{n} = \binom{2n-2}{n} + \binom{2n-2}{n-1}, \quad (4)$$

we see that this set of n -tuples decomposes into the disjoint union of a set consisting of $\binom{2n-2}{n} = \binom{2n-2}{n-2}$ n -tuples and a set consisting of $\binom{2n-2}{n-1}$ n -tuples. Denote these sets by P_1 and P_2 , respectively. For an arbitrary but fixed hyperplane H , they can be chosen so that the n -tuples in P_2 contain, but the n -tuples in P_1 do not contain H . Let L_1 be the set of $(n-1)$ -tuples obtained from the n -tuples of P_2 by omitting H from them. Let

$$\varphi : L_1 \rightarrow P_2 \quad (5)$$

be the bijection defined by this procedure in a natural way. Furthermore, let L_2 be the set complementary to L_1 in the set of $(n-1)$ -tuples of hyperplanes determining the lines of $DCD(n)$.

Thus, P_1 and L_1 consist of $\binom{2n-2}{n-2}$ n -tuples and $\binom{2n-2}{n-1}$ $(n-1)$ -tuples, respectively, such that they do not contain H . On the other hand, P_2 and L_2 consist of $\binom{2n-2}{n-1}$ n -tuples and $\binom{2n-2}{n-2}$ $(n-1)$ -tuples, respectively, each containing H .

Hence, P_1 and L_1 yield a set of points and a set of lines, respectively, with cardinalities corresponding to type (2); the same is valid in the case of P_2 and L_2 , regarding type (3). Besides, from each of the n -tuples in P_1 one can obviously omit precisely n distinct hyperplanes so as to obtain an $(n-1)$ -tuple; thus, each point determined by these n -tuples is incident with precisely n lines determined by the $(n-1)$ -tuples obtained in this way. Note that, since the n -tuples do not contain H , neither do the $(n-1)$ -tuples; hence the set of all these latter is just equal to L_1 . Furthermore, each of these $(n-1)$ -tuples can be completed by a hyperplane other than H in precisely $(n-1)$ distinct ways. Thus we obtain that the points and lines yielded by the sets P_1 and L_1 , respectively, form a configuration \mathcal{C}_1 of the desired type (2). By similar reasoning, one sees that a configuration \mathcal{C}_2 of type (3) can be obtained from the sets P_2 and L_2 .

The pairs of points and lines determined by the n -tuples and $(n-1)$ -tuples, respectively, which correspond to each other under the bijection φ in (5) form the set I of (3).

Now we define the map $\delta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, as follows. For any point p in \mathcal{C}_1 determined by an n -tuple of hyperplanes, take the line l in \mathcal{C}_2 determined by the complementary $(n-1)$ -tuple, and set $\delta(p) = l$. Likewise, for any line m in \mathcal{C}_1 determined by an $(n-1)$ -tuple of hyperplanes, take the point q in \mathcal{C}_2 determined by the complementary n -tuple, and set $\delta(m) = q$. De Morgan's laws imply that δ is a duality map. The symmetry of the construction which does not depend on the order of the elements in n -tuples enables us to conclude that both configurations are flag-transitive. \blacksquare

Here we emphasize again the fact that has already been touched following Theorem 2.2.

Remark 4.2. The duality map δ just defined cannot be extended to include, in the particular case of $n = 4$, the Pascal lines and Kirkman points as well. For, although they form a configuration (60_3) [27, 28], this configuration is not combinatorially self-polar [26, 9].

Corollary 4.3. *The configuration $DCD(n)$ is self-polar.*

Proof. The statement already follows from Theorems 3.4 and 3.3. A self-polarity map π can now be realized as follows. Consider a decomposition

$DCD(n) = \mathcal{C}_1 \oplus_I \mathcal{C}_2$ as in Theorem 4.1, and take the duality map $\delta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$. Then

$$\pi(x) = \begin{cases} \delta(x) & \text{if } x \text{ is a point or line in } \mathcal{C}_1, \\ \delta^{-1}(x) & \text{if } x \text{ is a point or line in } \mathcal{C}_2. \end{cases}$$

■

In the particular case of $n = 3$, the decomposition theorem 4.1 leads to the following rephrasing of Desargues' theorem.

Theorem 4.4. *The following statement is equivalent to Desargues' theorem. Given a complete quadrangle Q , one can choose six points, one on each line of Q , such that these points determine a complete quadrilateral.*

Proof. First we show that Desargues' theorem implies the statement. Consider the Desargues configuration, in which a complete quadrangle Q is chosen, as it is depicted in Figure 2. Note that the centre of perspectivity is the white point not incident with the shaded triangle of Q . Let Q' denote the desired quadrilateral. For finding the vertices of Q' , first choose the three black points which are vertices of the other shaded triangle. Then we choose three more points, such that each is formed as the intersection of the lines of the two triangles corresponding to each other under central perspective. But these latter three points are collinear by Desargues' theorem, so the six points altogether form indeed a complete quadrangle Q' .

The converse implication can simply be obtained by taking the complete quadrangle and the complete quadrilateral as in Figure 2, and choosing the shaded triangles as corresponding to each other under perspectivity. ■

Note that likewise to Desargues' theorem, the dual of our equivalent statement also holds.

In what follows we apply the decomposition theorem 4.1 to Danzer's configuration (35_4) .

Lemma 4.5. *The Steiner-Plücker configuration $(20_3, 15_4)$ in Theorem 2.1 can be obtained in the following way. Take seven hyperplanes, H_0, \dots, H_6 in general position in the 4-dimensional projective space. Take all the quadruples and triples such that each contain, say, H_0 . The intersections of the hyperplanes in these subsets yield 20 points and 15 lines, respectively, which, by suitable projection into the projective plane, form the desired configuration.*

Proof. See the labelling in Figure 4, which refers to the quadruples of hyperplanes determining the points. The intersection of the labels of points incident with the same line corresponds to the triple of hyperplanes determining that line. ■

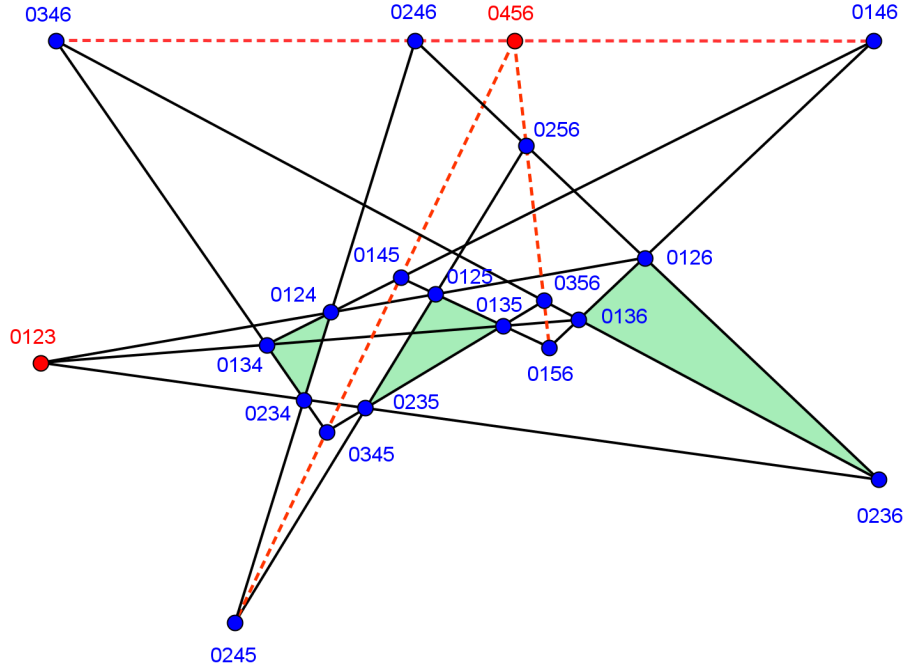


Figure 4: The Steiner-Plücker configuration $(20_3, 15_4)$. The points are labelled in accordance with Lemma 4.5. The shaded triangles are perspective from the same point (labelled by 0123), as in Theorem 2.1; the point of concurrency of the three axes of perspectivity is labelled by 0456.

Thus, together with that arising from Theorem 2.1, one can see here three different ways of deriving the Steiner-Plücker configuration. We remark that Adler gives a fourth one, see [1]. Independently this configuration is also presented by B. Servatius and H. Servatius [34] under the name of *generalized Reye configuration*. It is possible to verify that our Figure 4 depicts the same configuration as that in Figure 5 of [34]; see also Figure 6.50 in [28].

We also note the following interesting representation of this configuration. Lajos Szilassi observed [35] that in addition to the original three triangles there is another triple of triangles which seems to be in some definite relationship with the former. Using the labelling in our Figure 4, it readily follows that there is a combinatorial automorphism φ of the configuration which, in terms of the defining hyperplanes in Lemma 4.5, can be given in

the form

$$\begin{aligned} 1 &\longleftrightarrow 6 \\ 2 &\longleftrightarrow 5 \\ 3 &\longleftrightarrow 4. \end{aligned}$$

In general, φ cannot be realized geometrically (in the most general case, by a projective collineation of period two, i.e. by a *harmonic homology* [12]); in other words, given a particular representation of the Steiner-Plücker configuration, there is no geometric transformation carrying it into itself that would induce φ . However, we found that there exist even centrally symmetric representations. Figure 5 shows such a representation, which is particularly pleasing in the sense that Theorem 2.1 can easily be read out from it.

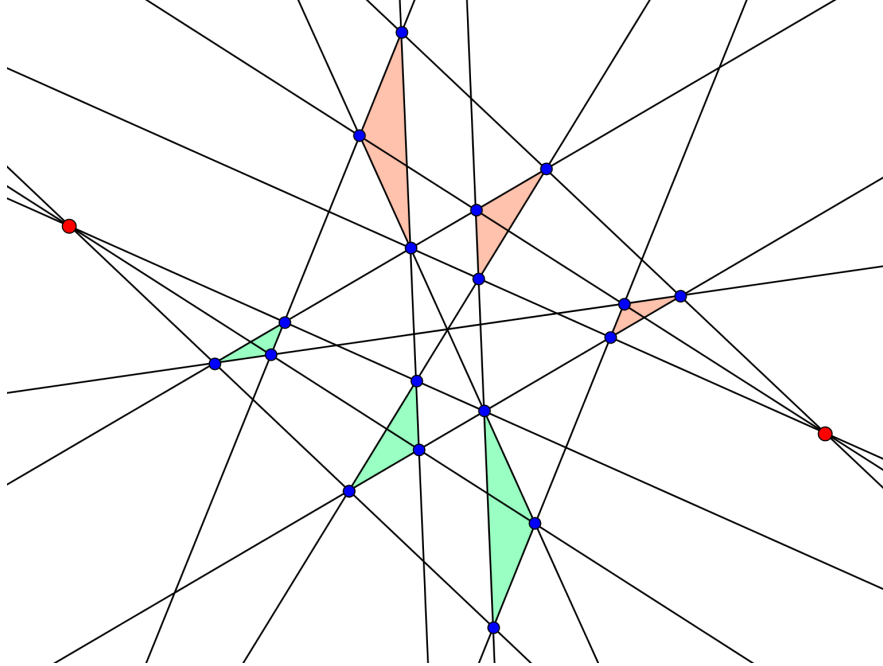


Figure 5: A centrally symmetric realization of the Steiner-Plücker configuration $(20_3, 15_4)$.

Lemma 4.6. *The Cayley-Salmon configuration $(15_4, 20_3)$ in Theorem 2.2 can be obtained in the following way. Take six hyperplanes, H_1, \dots, H_6 , in general position in the 4-dimensional projective space. They meet by fours and by threes in 15 points and 20 lines, respectively. The configuration formed*

by these points and lines is then to be suitably projected into the projective plane.

Proof. Since the Steiner-Plücker configuration and the Cayley-Salmon configuration are dual to each other, the statement is a straightforward consequence of Theorem 4.1 and Lemma 4.5. ■

Note that similarly to the proof of the previous lemma, this latter can also be inferred from the labelling of Figure 6, which depicts the Cayley-Salmon configuration.

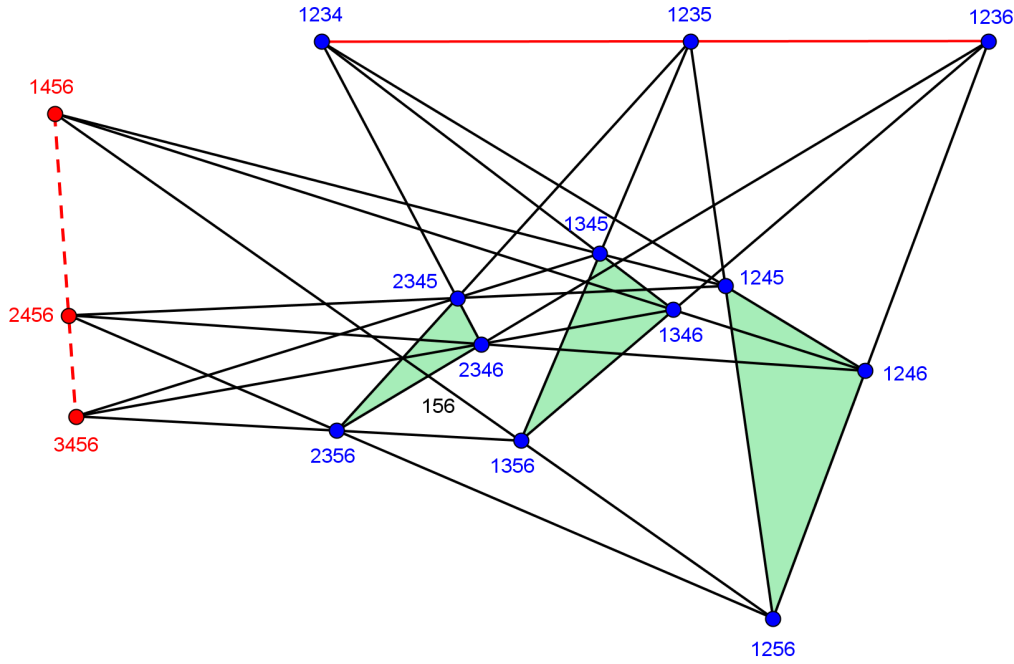


Figure 6: The Cayley-Salmon configuration $(15_4, 20_3)$. The points are labelled in accordance with Lemma 4.6; they also indicate the duality with the Steiner-Plücker configuration depicted in Figure 4, as it is described in the proof of Theorem 4.1.

Theorem 4.7. *Danzer's configuration (35_4) can be decomposed into the incidence sum of the Steiner-Plücker configuration $(20_3, 15_4)$ and the Cayley-Salmon configuration $(15_4, 20_3)$.*

Proof. The statement follows from Theorem 4.1, by Lemma 4.5 and Lemma 4.6. ■

Remark 4.8. A comparison of Theorems 2.3 and 4.7 shows that Leopold Klug knew the configuration (35_4) what we call Danzer’s configuration (to be precise, we traced it back to this point; cf. Remark (vi) of Grünbaum in his “Musings” [21]).

Remark 4.9. Potočník proved [29] that there is exactly one biregular graph with 20 vertices of valence 3 and 15 vertices of valence 4 that is edge-transitive of girth 6. In his Table 2 it has $ID = \{35, 2\}$. One may conclude that this graph is the Levi graph of both the Cayley-Salmon and the Steiner-Plücker configuration. From [31] it follows that there are only two flag-transitive self-polar combinatorial (35_4) configurations. From [37] we may deduce that the other one is a cyclic configuration with the symbol $\{0, 1, 8, 14\}$. In particular, this means its Levi graph is a cyclic Haar graph, see [24], and is denoted as $C4[70, 3]$ in [37]. The *Danzer graph*, i.e. the Levi graph of Danzer’s configuration is denoted as $C4[70, 4]$. It is the Kronecker cover over the Odd graph O_4 (cf. Theorem 3.2 and Corollary 3.5).

We refer the reader to an interesting paper [3] in which further information about the nature of both configurations can be read off the Table 4 for $v = 35$. In that table $C4[70, 3]$ is immediately followed by $C4[70, 4]$.

5 Another subconfiguration of Danzer’s configuration

Here we show that besides those given in Theorem 4.7 there is a further interesting subconfiguration of Danzer’s (35_4) . Actually, we shall make use of it in the next section.

Consider the Coxeter graph, which is a 28-vertex cubic graph. A particularly nice drawing of it, due to Milan Randić [5], is shown in Figure 7a. The vertices can be identified with the 28 antiflags, i.e. non-incident point-line pairs, of the Fano plane \mathcal{F} . Two vertices are adjacent if and only if the corresponding members in pairs of distinct antiflags are also non-incident [18]. This is equivalent to saying that the vertices are labelled with triplets of points in \mathcal{F} which do not lie on the same line, and adjacent vertices are labelled with disjoint triplets. It follows that the Coxeter graph is a subgraph of the Kneser graph $K(7, 3)$ [19].

A point-line realization of the combinatorial configuration (28_3) obtained from the Coxeter graph Γ by V -construction is shown in Figure 7b. We denote it by $N(\Gamma)$ and name it the *Coxeter (28_3) -configuration*.

Remark 5.1. Sometimes a configuration of type (12_3) that has the Nauru graph, i.e. the arc-transitive generalized Petersen graph on 24 vertices, as its

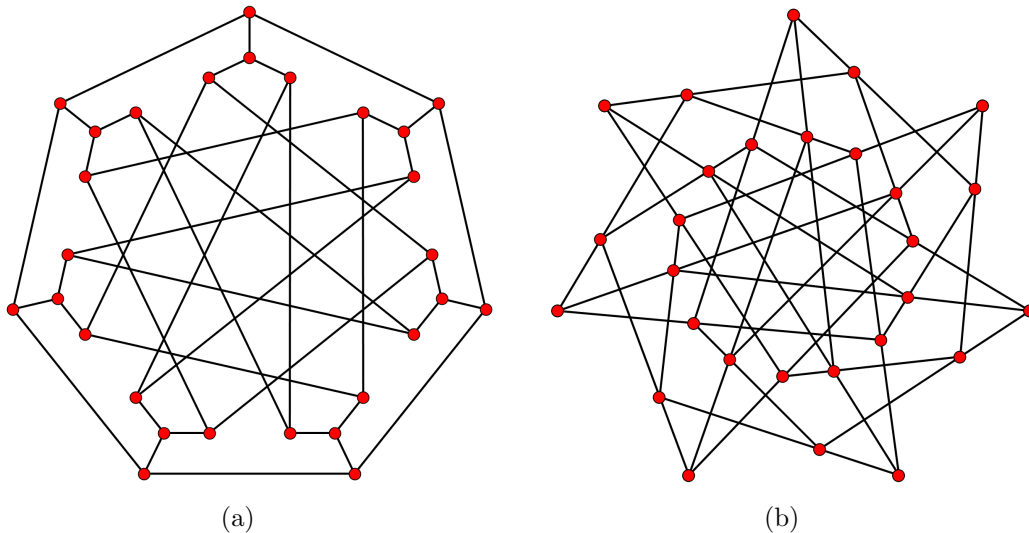


Figure 7: The Coxeter graph (a), and the corresponding point-line Coxeter (28_3) -configuration (b), both with 7-fold rotational symmetry.

Levi graph, is called the Coxeter configuration, instead of Nauru configuration. If we want to make distinction between the two, we propose that the latter be called Coxeter (12_3) -configuration.

The representation exhibiting seven-fold rotational symmetry makes it easy to understand the structure of both Γ and $N(\Gamma)$. In both cases we have three regular heptagons, one convex and two star-heptagons; the latter are of type $\{7/2\}$ and $\{7/3\}$ (here we adopt the notation of star-polygons used by Coxeter [10, 11]). The set of points is obviously the same in both structures; and the vertices of each regular heptagon form an orbit of the cyclic group of order 7. There is one additional seven-point orbit in both cases. The i th point v of this orbit is adjacent with the i th vertex of each heptagon in Γ ; accordingly, the i th line of each heptagon in $N(\Gamma)$ is incident with the corresponding i th point of the non-heptagonal orbit. Besides the three line-orbits of the heptagons in $N(\Gamma)$, there is a fourth line-orbit, whose i th line corresponds to the vertex-neighbourhood of v in Γ . Thus the i th line of this line-orbit in $N(\Gamma)$ is incident with the i th point of each heptagonal orbit. Furthermore, in accordance with the self-polarity of $N(\Gamma)$ (by Theorem 3.3), the non-heptagonal point-orbit is the polar of the non-heptagonal line-orbit.

Based on these relationships, one might expect that the rotational realization of $N(\Gamma)$ could be extended to a heptagonally symmetric realization of the Danzer's configuration. For, a comparison of our observations above with the representation of the Odd graph O_4 depicted in Figure 1 suggest

that $N(\Gamma)$ can be extended, by adding a new seven-point orbit (together with corresponding edges), to a representation of O_4 exhibiting the dihedral symmetry D_7 . This can be performed by using either a central or an axial reflection, which leaves invariant the heptagonal orbits, but doubles the non-heptagonal orbit.

In [21] Grünbaum admits he unsuccessfully tried to find a symmetric drawing of Danzer's configuration. Several of our attempts, too, aimed at obtaining such a highly symmetric point-line representation of Danzer's configuration, failed. Throughout these experiments, we exploited the property that $N(\Gamma)$ is movable; in fact, the relative size and position of the orbits can be changed (while keeping the rotational symmetry). (For a formal definition of movability of a configuration, see [22]; cf. also [17].) For instance, Figure 8 shows a realization of the configuration (28_3) different from that in Figure 7a.

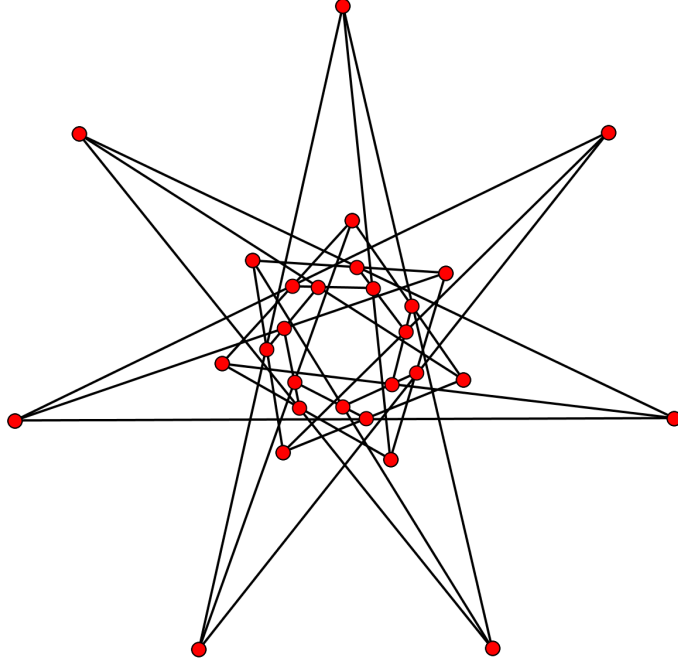


Figure 8: Another realization of the point-line Coxeter configuration (28_3) with 7-fold rotational symmetry.

Our experiences suggest the following conjecture (cf. also Construction 2 and 3 in the next section).

Conjecture 5.2. *There is no realization of Danzer's (35_4) point-line configuration with five- or seven-fold rotational symmetry.*

We remark that this is in accordance with a similar property of Desargues' (10_3) point-line configuration, which can be proved [17]. One may also ask whether such a property of non-realizability of $DCD(n)$ with n -fold symmetry is valid for all n . We did not investigate this problem in such generality; even verification of the conjecture for $n = 4$ needs further work (possibly with detailed calculations), which is beyond the scope of the present paper.

6 Some constructions for Danzer's configuration

CONSTRUCTION 1.

This construction is based on Theorem 4.7. Start from a concrete representation \mathcal{C}_1 of the Steiner-Plücker configuration $(20_3, 15_4)$, e.g. from that depicted in Figure 4. Choose the complete quadrangle determined by the vertices 0123, 0124, 0134, 0234. Take a complete quadrilateral determined by four new lines such that they are labelled by 456, 356, 256, 156, and they are incident with the vertices 0456, 0356, 0256, 0156, respectively. By a basic theorem of projective geometry [12], there is a unique projective correlation sending the vertices of the quadrangle into the lines of the corresponding quadrilateral. In our case this correlation is just the duality map δ defined in the proof of Theorem 4.1, and determined by the labels chosen here. Accordingly, the complete quadrilateral can be extended to a representation \mathcal{C}_2 of the Cayley-Salmon configuration $(15_4, 20_3)$. The incidence sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ yields the Danzer's configuration.

In what follows we shall need some old theorems of elementary geometry, due to Jacob Steiner, 1828 [13].

Theorem 6.1. *Suppose four lines are given in general position, i.e. they intersect two by two at six points.*

- (a) *These four lines, taken three by three, form four triangles whose circum-circles pass through the same point F .*
- (b) *The centres of the four circles lie on the same circle.*
- (c) *The orthocentres of the four triangles lie on the same line.*

We note that Theorem 6.1.a is also known as Wallace's theorem, since it was first stated by William Wallace, 1806 [8].

CONSTRUCTION 2.

We start from 7 lines in general position, i.e. no three of them meet in a point. Each triple of these lines determines a triangle, of which we take the orthocentre. Thus we have $\binom{7}{3} = 35$ points, such that they are labelled by a 3-element subset of a 7-element set. Taking the lines four by four, each such quadruple determines a line, as in the conclusion of Theorem 6.1.c. Thus we have $\binom{7}{4} = 35$ lines, which are labelled by a 4-element subset of the same 7-element set as before. Furthermore, each of the orthocentres belongs to four distinct quadruples of orthocentres determining a line by Theorem 6.1.c; thus each of the 35 points is incident with four of the 35 lines. It follows in turn that each line is incident with four points. Hence we have the desired (35_4) configuration.

One might expect that an arrangement of seven lines with D_7 symmetry would yield, via the construction just described, a representation of this configuration with the same symmetry. However, it is found that certain coincidences of lines exclude such a possibility. This experience also supports our Conjecture 5.2.

Thus, our attempts to obtain highly symmetric representation of Danzer's (35_4) configuration failed; moreover, one cannot expect that Construction 1 would yield a visually attractive representation at all. Hence, to obtain a graphical representation, we use eventually the following construction.

CONSTRUCTION 3.

In [4] a concept of *polycyclic configurations* was introduced. A configuration is polycyclic if there exist a non-trivial cyclic automorphism α which is semi-regular (all orbits of points and lines are of the same size). If α has order k , then such configuration is called k -cyclic. If a realization of a k -cyclic configuration in the Euclidean plane realizes α as a rotation for angle $2\pi/k$, then such realization (if exists) is called *rotational*. A theory was developed in [4] to analyze polycyclic configurations via voltage graphs (also called the reduced Levi graphs in [22]; for more details on voltage graphs, see e.g. [20]).

Danzer's configuration is 5-cyclic and 7-cyclic. The automorphism group of its Levi graph is $S_7 \times \mathbb{Z}_2$, the direct product of the symmetric group of degree 7 with the group of order two; hence its order is 10080. The Levi graph is, for example, a \mathbb{Z}_5 covering graph over the voltage graph in Figure 9a, and a \mathbb{Z}_7 covering graph over the voltage graph in Figure 9b. To obtain rotational realization it is necessary to find solutions of polynomial equations which depend directly on the voltage graph structure. Using MATHEMATICA we have analyzed all possible voltage graphs for 5- and 7-cyclic structures, but we

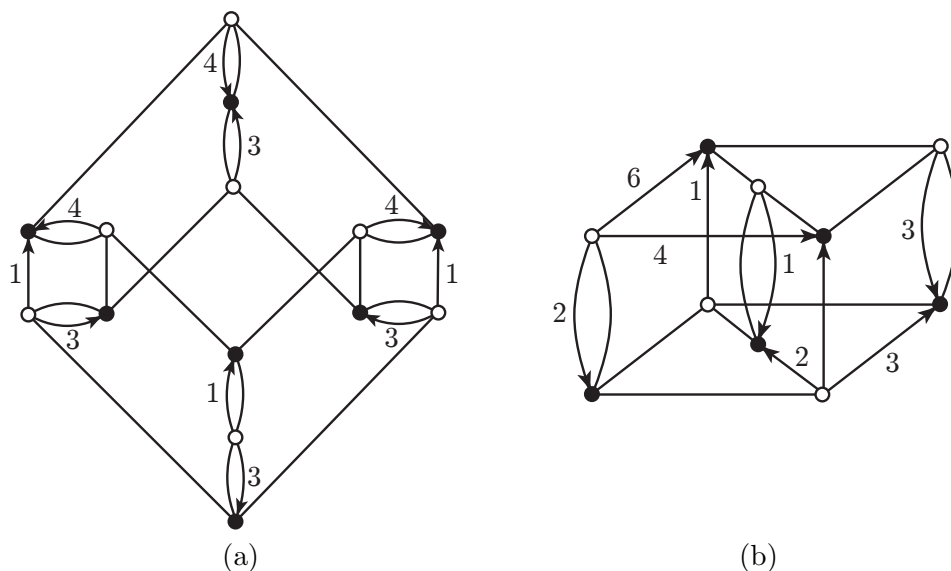


Figure 9: Voltage graphs for polycyclic representation of Danzer's configuration; 5-cyclic (a), and 7-cyclic (b).

were not able to find any solution of the corresponding equations which would produce a rotational realization. The SOLVE method either failed to give any solutions, returned complex solutions, or the solutions led to representations where points (or lines) of different orbits coincide. For example, the voltage graph in Figure 9b gives representation shown in Figure 10 where two line-orbits coincide. However, it is possible to “perturb” points slightly to obtain a realization in the plane which still resembles the 7-cyclic symmetry, see Figure 11. Following Grünbaum [21], we could try to substitute the lines by pseudolines so that hopefully the cyclic (but not dihedral) symmetry is preserved.

We note in passing that we only found two bipartite voltage graphs for the Danzer graph. In [37] three minimal voltage graphs are presented: one on five vertices and two on seven vertices. They are shown in Figure 12.

Finally, we note that Klug also describes a construction in his book ([26], Chapter 8). It is based essentially on Theorem 2.4. Our aim in this Section was to find new, independent ways (especially, with regard to the question of symmetry).

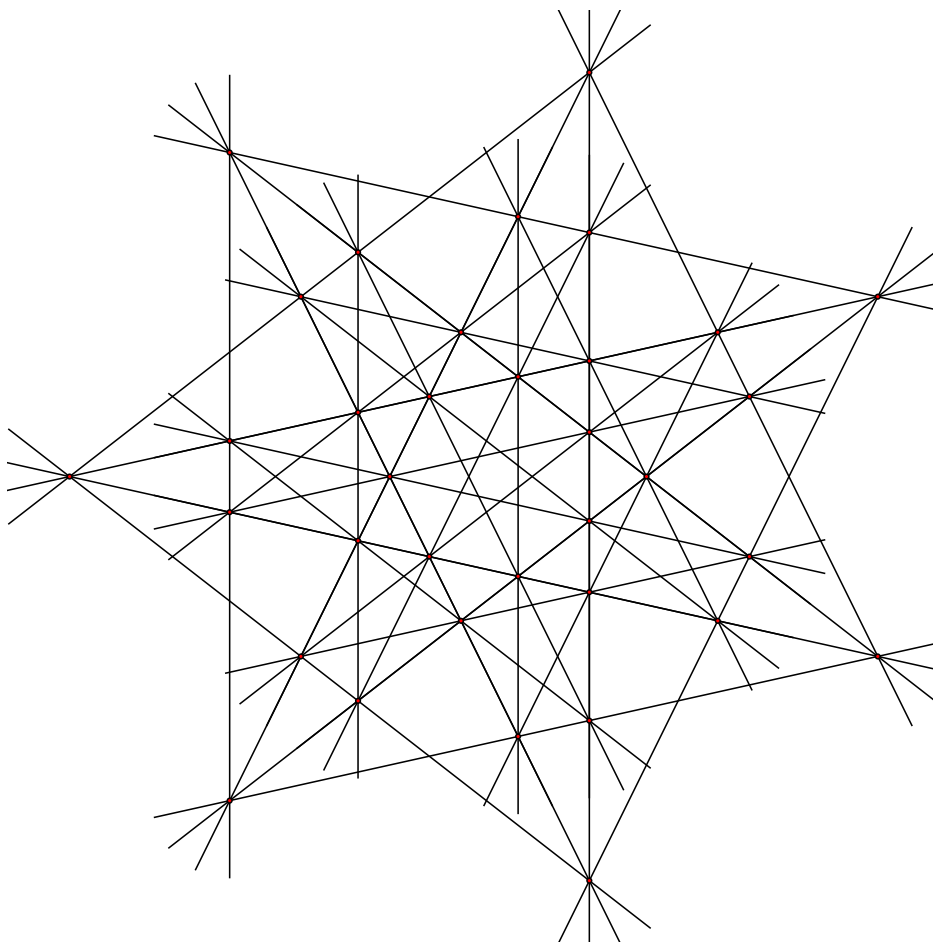


Figure 10: Representation of Danzer's configuration with 7-fold dihedral symmetry, where lines from two different orbits coincide.

7 Point-circle representations

The representation of the graph O_4 given in Figure 1 has the property that each vertex-neighbourhood forms a concyclic set, i.e. a circle can be drawn through these points. Hence, the V -construction yields directly a point-circle configuration [17]. This configuration is shown in Figure 13. Note that this realization of Danzer's configuration exhibits the dihedral D_7 symmetry; thus, this symmetry, which is the maximal possible in dimension two, can easily be achieved in this case, in contrast to the point-line realization (cf. Conjecture 5.2). We know that this is just a particular case of the following more general result [17].

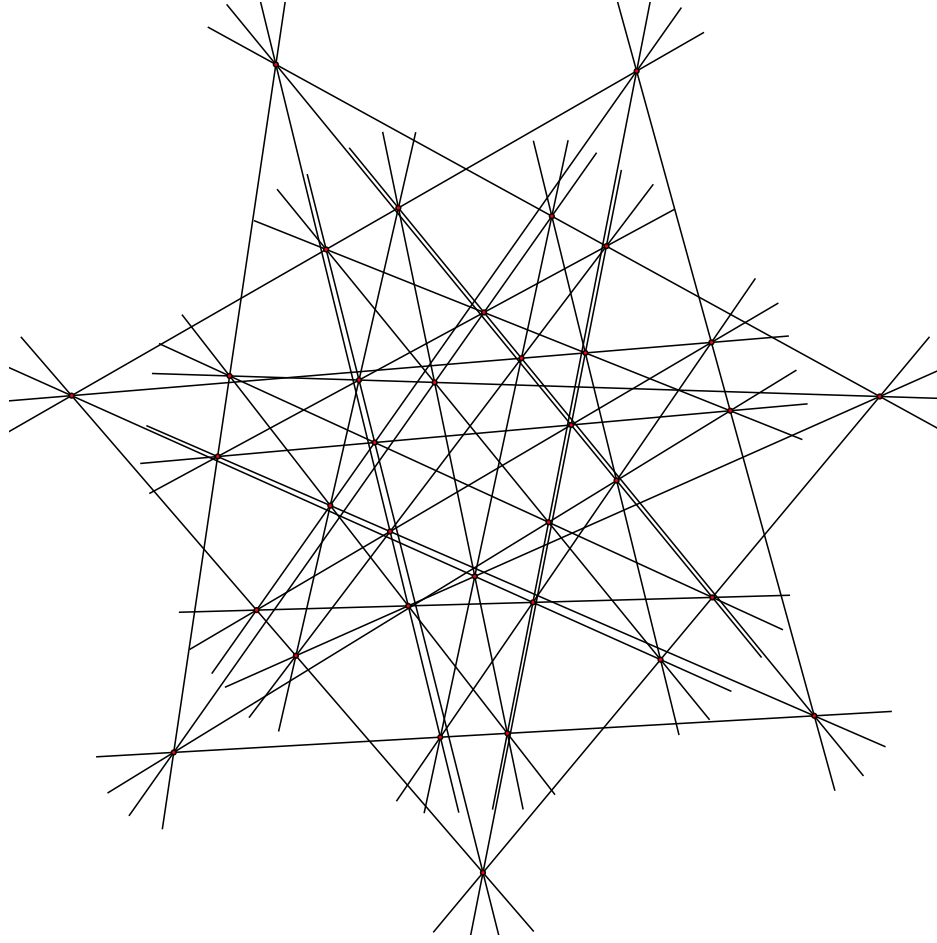


Figure 11: Realization of Danzer's configuration resembling the 7-fold rotational symmetry.

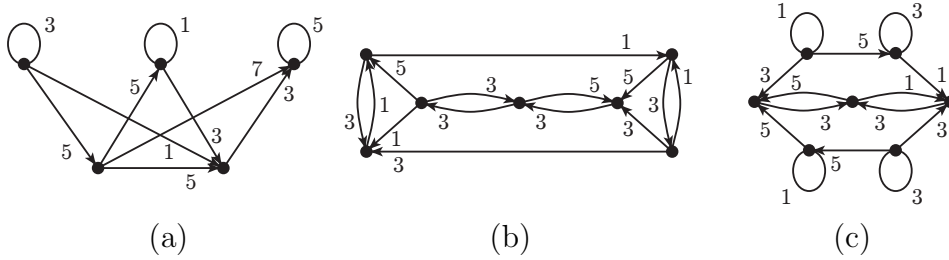


Figure 12: Voltage graphs for the Danzer graph following [37]. The Danzer graph is a \mathbb{Z}_{14} -covering graph over (a) and a \mathbb{Z}_{10} -covering graph over (b) and (c).

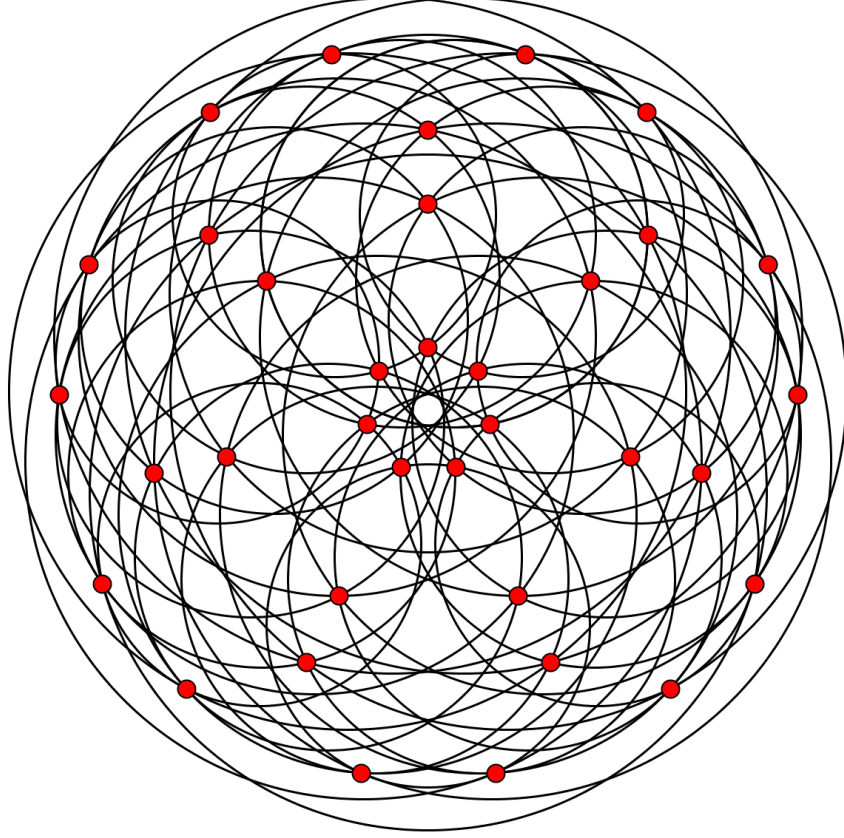


Figure 13: A point-circle realization of Danzer's configuration with 7-fold dihedral symmetry.

Theorem 7.1. *For all $n \geq 3$, there exists an isometric point-circle configuration of type*

$$\left(\binom{2n-1}{n-1}_n \right).$$

It can be obtained from the Odd graph O_n by V-construction.

In other words, for all $n \geq 2$, $DCD(n)$ can be realized as an (isometric) point-circle configuration. Moreover, it is also shown in [17] that each member of this infinite series of configurations forms a subconfiguration of certain members of Clifford's renowned infinite series of configurations. (Here we recall that a point-circle configuration is called *isometric* if all the circles are of the same size [17].) Note that our example presented in Figure 13 is not isometric; however, as it is movable, its shape can be changed continuously while keeping the D_7 symmetry.

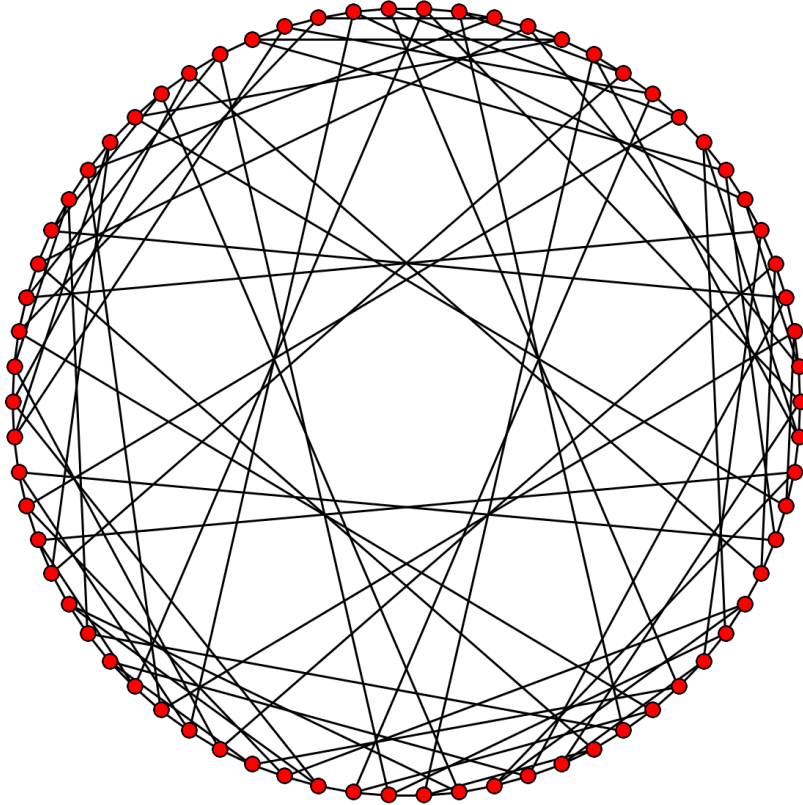


Figure 14: Hamiltonian representation of the Danzer graph exhibiting 5-fold dihedral symmetry.

Figure 14 exhibiting the Hamilton cycle of the Danzer graph shows that a representation of this graph with D_5 symmetry is possible. Thus, one can expect that by suitably modifying this representation and applying the V -construction, a point-circle representation of Danzer's configuration can be realized with D_5 symmetry. Work in this direction is still in progress.

In what follows we present two further constructions for the point-circle realization of Danzer's configuration. They are based on Steiner's theorems given in the preceding section (Theorem 6.1).

In our first construction, we use Theorem 6.1.a. We start from 7 lines in general position, i.e. no three of them meet in a point. Each triple of these lines determines a triangle, of which we take the circumcircle. Taking the lines four by four, each such quadruple determines a point, as in the conclusion of Theorem 6.1.a. We obtain in this way altogether $\binom{7}{4} = 35$ points. There are $\binom{7}{3} = 35$ circles, all being circumcircles as determined above. Thus the points are labelled by the 4-element subsets of a 7-element set, and the

circles are labelled by the 3-element subsets of the same set. Furthermore, each circle belongs to four distinct quadruples of circles determining a point by Theorem 6.1.a; thus each circle is incident with four points. It follows in turn that each point is incident with four circles (since the number of points and circles is the same). Hence we have the desired point-circle configuration.

Using the same arguments, it is directly seen that Theorem 6.1.b provides a way of construction for an isomorphic point-circle configuration (35_4) . Here the points are labelled by the 3-element subsets of the 7-element set of lines, and the circles (which are determined as in the conclusion of Theorem 6.1.b) are labelled by the 4-element subsets of the same set.

The three subconfigurations of Danzer's configuration considered in the previous sections also have point-circle realizations. For the Steiner-Plücker and the Cayley-Salmon configuration these can be obtained by a simple deletion procedure from a suitable version of that shown in Figure 13; the result is depicted in Figure 15.

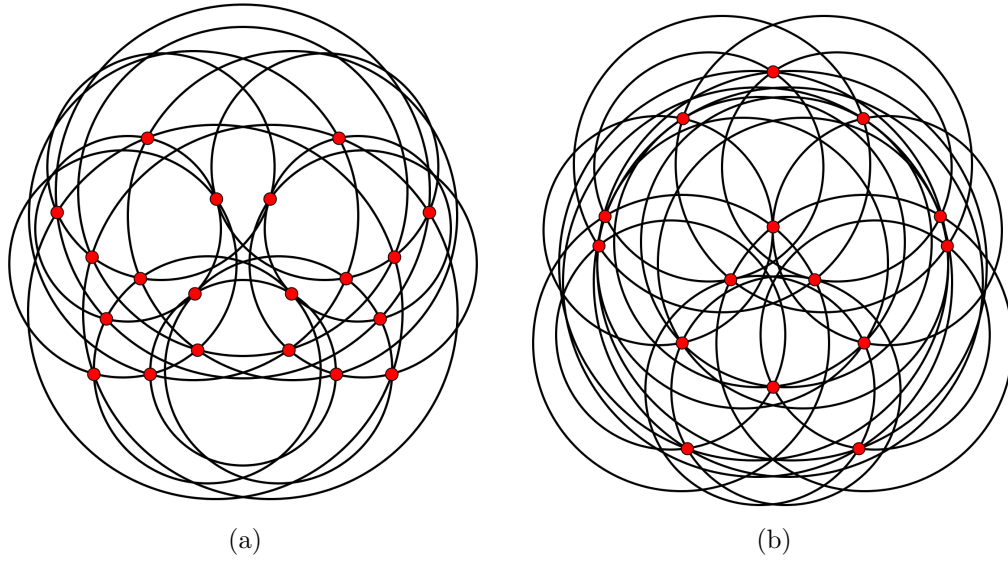


Figure 15: Point-circle realization of the $(20_3, 15_4)$ Steiner-Plücker configuration (a), and the $(15_4, 20_3)$ Cayley-Salmon configuration (b), both with vertical mirror symmetry.

Finally, a point-circle realization of the configuration (28_3) obtained from the Coxeter graph by V-construction is shown in Figure 16. Here we use a unit-distance representation of the Coxeter graph, due to Gerbracht [14] (we recall that a drawing of a graph G is called a *unit-distance representation* of G if all edges in the drawing have the same length [38]). Consequently, the

configuration is isometric (for isometric point-circle configurations obtained from unit-distance graphs by V -construction, see [17]).

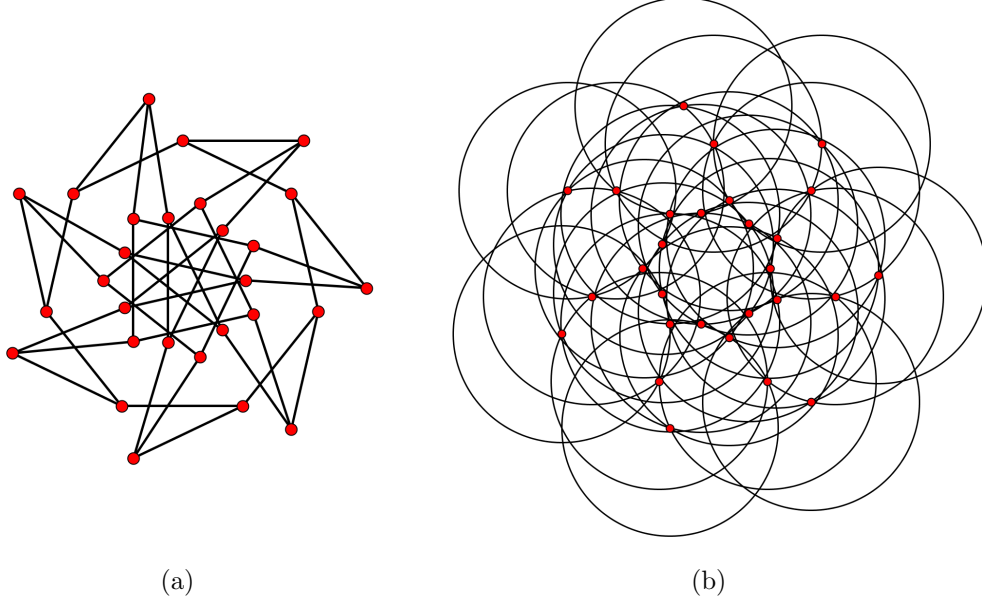


Figure 16: A unit-distance representation of the Coxeter graph (a), and the isometric point-circle configuration (28_3) derived from it (b), both with 7-fold rotational symmetry.

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References

- [1] V. E. Adler, Some incidence theorems and integrable discrete equations, *Discrete Comput. Geom.* **36** (2006), 489–498.
- [2] H. F. Baker, *Principles of Geometry*, Vol. II, Cambridge University Press, Cambridge, 1930.

- [3] M. Boben, Š. Miklavič and P. Potočnik, Rotary polygons and configurations, *Electronic J. Combin.* **18** (2011), #P119.
- [4] M. Boben and T. Pisanski, Polycyclic configurations, *European J. Combin.* **24** (2003), 431–457.
- [5] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.
- [6] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2011.
- [7] W. B. Carver, On the Cayley-Veronese class of configurations, *Trans. Amer. Math. Soc.* **6** (1905), 534–545.
- [8] J. W. Clawson, The complete quadrilateral, *Ann. Math.* **20** (1905), 232–261.
- [9] J. Conway and A. Ryba, The Pascal Mysticum demystified, *Math. Intelligencer* **34** (2012), 4–8.
- [10] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455. Reprinted in: H. S. M. Coxeter, *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, 1968, 106–149.
- [11] H. S. M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1961.
- [12] H. S. M. Coxeter, *Projective Geometry*, University of Toronto Press, 1974.
- [13] J.-P. Ehrmann, Steiner’s Theorems on the complete quadrilateral, *Forum Geom.* **4** (2004), 35–52.
- [14] E. H.-A. Gerbracht, *On the Unit Distance Embeddability of Connected Cubic Symmetric Graphs*, Kolloquium über Kombinatorik. Magdeburg, Germany. Nov. 15, 2008.
<http://mathworld.wolfram.com/CoxeterGraph.html>
- [15] G. Gévay, Symmetric configurations and the different levels of their symmetry, *Symmetry Cult. Sci.* **20** (2009), 309–329.
- [16] G. Gévay, Constructions for large spatial point-line (n_k) configurations, *Ars Math. Contemp.* (In press.)

- [17] G. Gévay and T. Pisanski, Kronecker covers, V -construction, unit-distance graphs and isometric point-circle configurations, *Ars Math. Contemp.* (In press.)
- [18] C. D. Godsil, Problems in algebraic combinatorics, *Electron. J. Combin.* **2** (1995), F1.
- [19] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
- [20] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.
- [21] B. Grünbaum, Musings on an example of Danzer's, *European J. Combin.* **29** (2008), 1910–1918.
- [22] B. Grünbaum, *Configurations of Points and Lines*, Graduate Texts in Mathematics, Vol. 103, American Mathematical Society, Providence, Rhode Island, 2009.
- [23] B. Grünbaum and J. F. Rigby, The real configuration (21_4) , *J. London Math. Soc.* **41** (1990), 336–346.
- [24] M. Hladnik, D. Marušič and T. Pisanski, Cyclic Haar graphs, *Discrete Math.* **244** (2002), 137–152.
- [25] W. Imrich and T. Pisanski, Multiple Kronecker covering graphs, *European J. Combin.* **29** (2008), 1116–1122.
- [26] L. Klug, *Az általános és négy különös Pascal-hatszög konfigurációja (The Configuration of the General and Four Special Pascal Hexagons; in Hungarian)*, Ajtai K. Albert Könyvnyomdája, Kolozsvár, 1898. Reprinted in: L. Klug, *Die Configuration Des Pascal'schen Sechseckes Im Allgemeinen Und in Vier Speciellen Fällen (German Edition)*, Nabu Press, 2010.
- [27] F. Levi, *Geometrische Konfigurationen*, Hirzel, Leipzig, 1929.
- [28] T. Pisanski and B. Servatius, *Configurations from a Graphical Viewpoint*, Birkhäuser Advanced Texts Basler Lehrbücher Series, Birkhäuser Boston Inc., Boston, 2013.
- [29] P. Potočník, Locally arc-transitive graphs of valence $\{3, 4\}$ with trivial edge kernel, to appear in *J. Alg. Combin.*

- [30] P. Potočník, A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index $(4,2)$, *European J. Combin.* **30** (2009), 1323–1336.
- [31] P. Potočník, P. Spiga and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *J. Symbolic Comp.* **50** (2013), 465–477.
- [32] P. Potočník and S. Wilson, Tetravalent edge-transitive graphs of girth at most 4, *J. Combin. Theory B.* **97** (2007), 217–236.
- [33] G. Salmon, *A Treatise on Conic Sections*, Chelsea Publ. Co., New York, 1954.
- [34] B. Servatius and H. Servatius, The generalized Reye configuration, *Ars Math. Contemp.* **3** (2010), 21–27.
- [35] L. Szilassi, personal communication.
- [36] O. Veblen and J. W. Young, *Projective Geometry*, Ginn and Company, Boston, 1910.
- [37] S. Wilson and P. Potočník, A Census of edge-transitive tetravalent graphs; Mini-Census.
<http://jan.ucc.nau.edu/~swilson/C4Site/index.html>
- [38] A. Žitnik, B. Horvat and T. Pisanski, All generalized Petersen graphs are unit-distance graphs, *J. Korean Math. Soc.* **49** (2012), 475–491.